CHALMERS
UNIVERSITY OF TECHNOLOGY

Small automorphic representations and degenerate Whittaker vectors

Henrik Gustafsson

Number Theory Seminar

Rutgers 2016

hgustafsson.se

Based on

Small automorphic representations and degenerate Whittaker vectors
HG, Axel Kleinschmidt, Daniel Persson
arXiv:1412.5625 [math.NT]
Submitted to Journal of Number Theory

Eisenstein series and automorphic representations
Philipp Fleig, HG, Axel Kleinschmidt, Daniel Persson
arXiv:1511.04265 [math.NT]
Submitted to Cambridge University Press

This talked is based on a paper together with AK & DP with the same title that we submitted to Journal of Number Theory a little over a year ago.

It also heavily leans on a review/book we submitted recently in collaboration with PF. It gives an overview of the theory of adelic automorphic forms along with the required background. It covers how to compute F coeffs and has a lot of examples, and interesting questions and applications for both mathematics and physics.

The topmost paper was started during the work on the review. It applies some of the tools described in there, to study the types of Fourier coefficients of interest in string theory.

Outline

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  $G(\mathbb{Q})\backslash G(\mathbb{A})/K$ | non-holomorphic Eisenstein series

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- Outlook

Motivation

There are many reasons for studying classical modular forms or automorphic forms and representations in both mathematics and physics.

In physics, automorphic forms are central in, for example string theory, in particular for computing scattering amplitudes and for BH microstate counting related to BH temperature

Recently, they have also figured in statistical mechanics for describing certain types of 2 dimensional crystals. [Brubaker-Bump-Friedberg, Baxter]
Let us focus on string scattering amplitudes.

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- Hecke eigenvalues
- Point counts of elliptic curves
- Langlands program
  L-functions | The Langlands–Shahidi method
- String theory
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  Two-dimensional models of crystals

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Let us focus on string scattering amplitudes.

String theory

As you know, instead of point particles and their trajectories, the fundamental objects of string theory are extended objects such as strings sweeping out world-sheets in spacetime.

We see that when taking the limit of the typical string length to 0 we end up with something that looks like a point particle

For historical reasons one often talks about the parameter alpha’ instead

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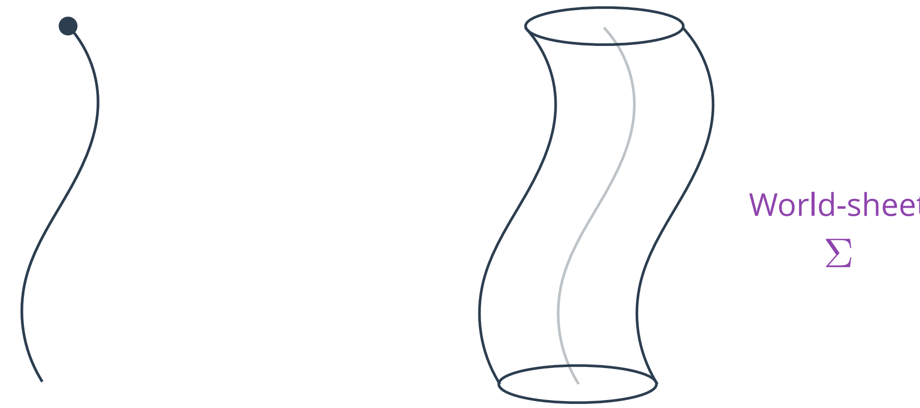

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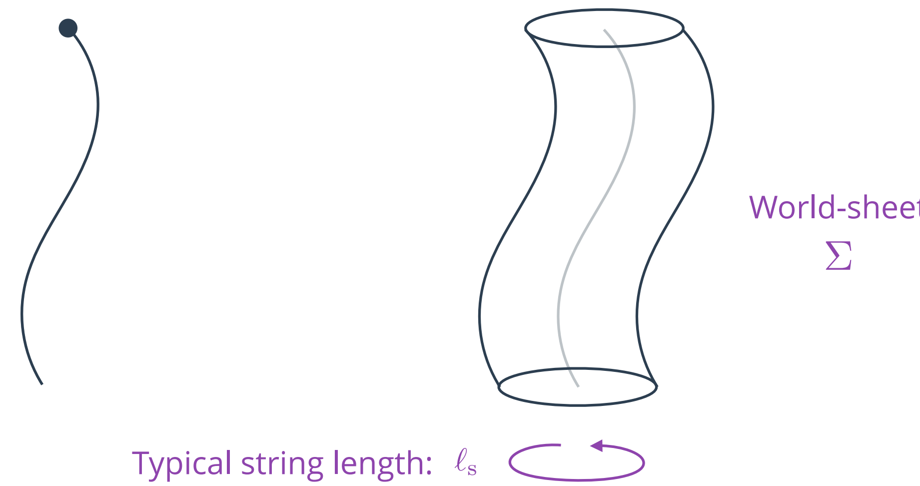

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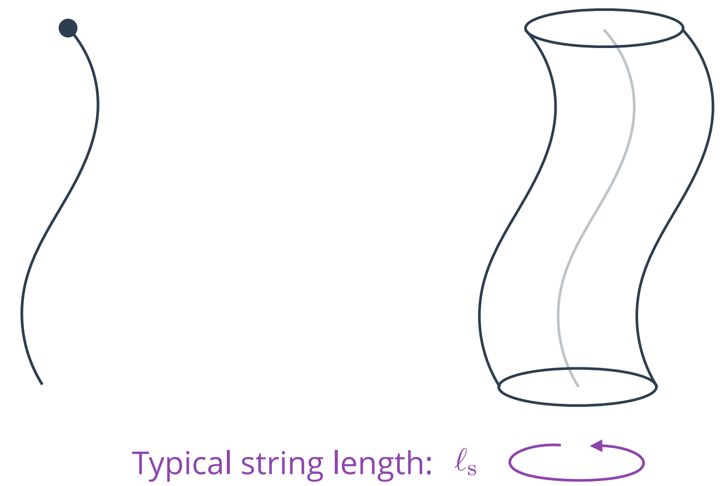

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Spacetime is described by a Riemannian manifold M

But to study physics in smaller dimensions one can compactify certain directions in M.
In this talk we will use toroidal compactifications - compactify on a torus - which preserves a lot of the symmetries we are interested in.

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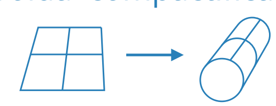

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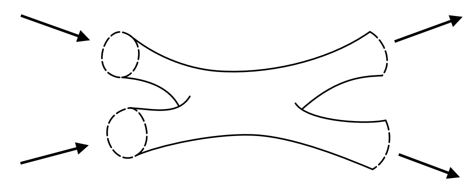

Strings interact by joining and splitting governed by the interaction strength: the string coupling gs

For example, this picture could describe the scattering of two gravitons coming in from the infinity.

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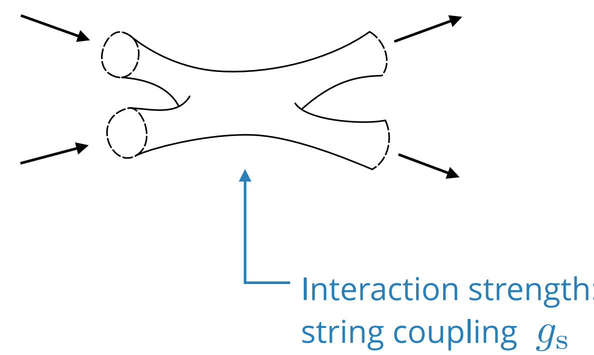

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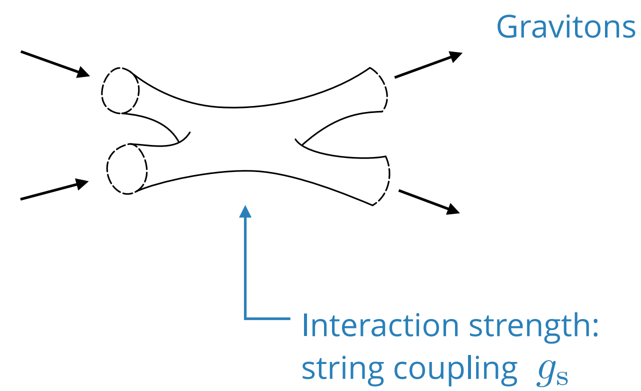

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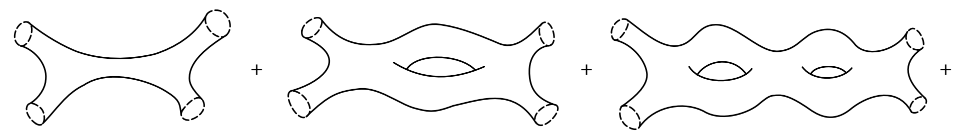
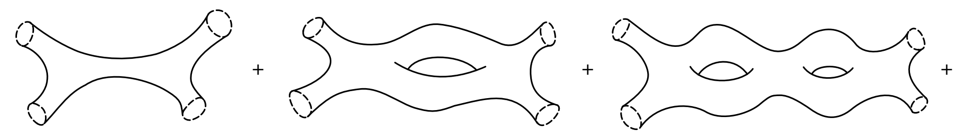


When computing the effects of interactions one has to sum over all the possible world-sheets including a sum over different topologies which are weighted by the string coupling to the power of minus the Euler characteristic giving us these different diagrams.

Interactions

Weighted by: $g_s^{-\chi_E}$ $\quad -\chi_E = 2(\text{genus} - 1) + \text{boundaries}$



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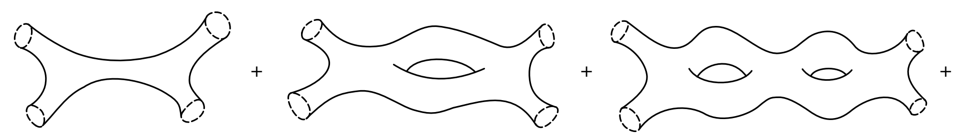

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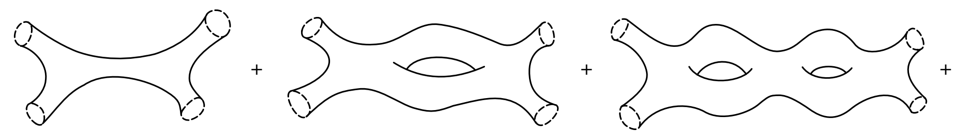

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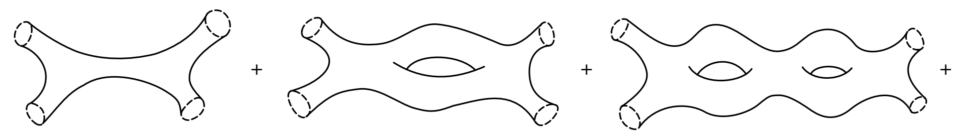

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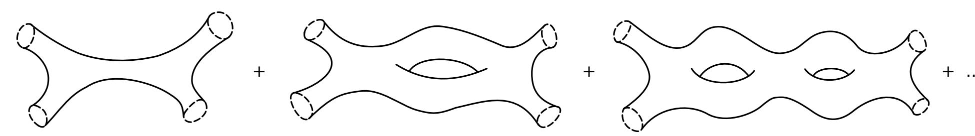

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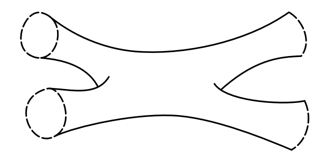
Gravitons  in *D* dimensions

Let us go back to the graviton in D dimensions.

The effect of the interactions can be described by the following Taylor expansion in alpha’. The first term described ordinary Einstein gravity (alpha’ -> 0 = point particles). The corrections are labeled by R4 D4R4 and D6R4 etc, which are known so called kinematic structure factors.

The interesting part for us though are the coefficients in front of these factors. Which we will now study and try to find.

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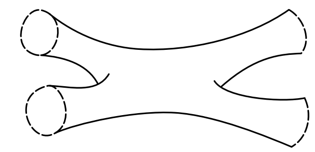
Gravitons  in *D* dimensions

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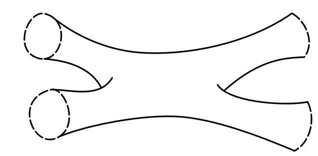
Gravitons  in *D* dimensions

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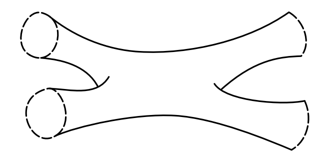
Gravitons  in *D* dimensions

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Kinematic structure factors (known)

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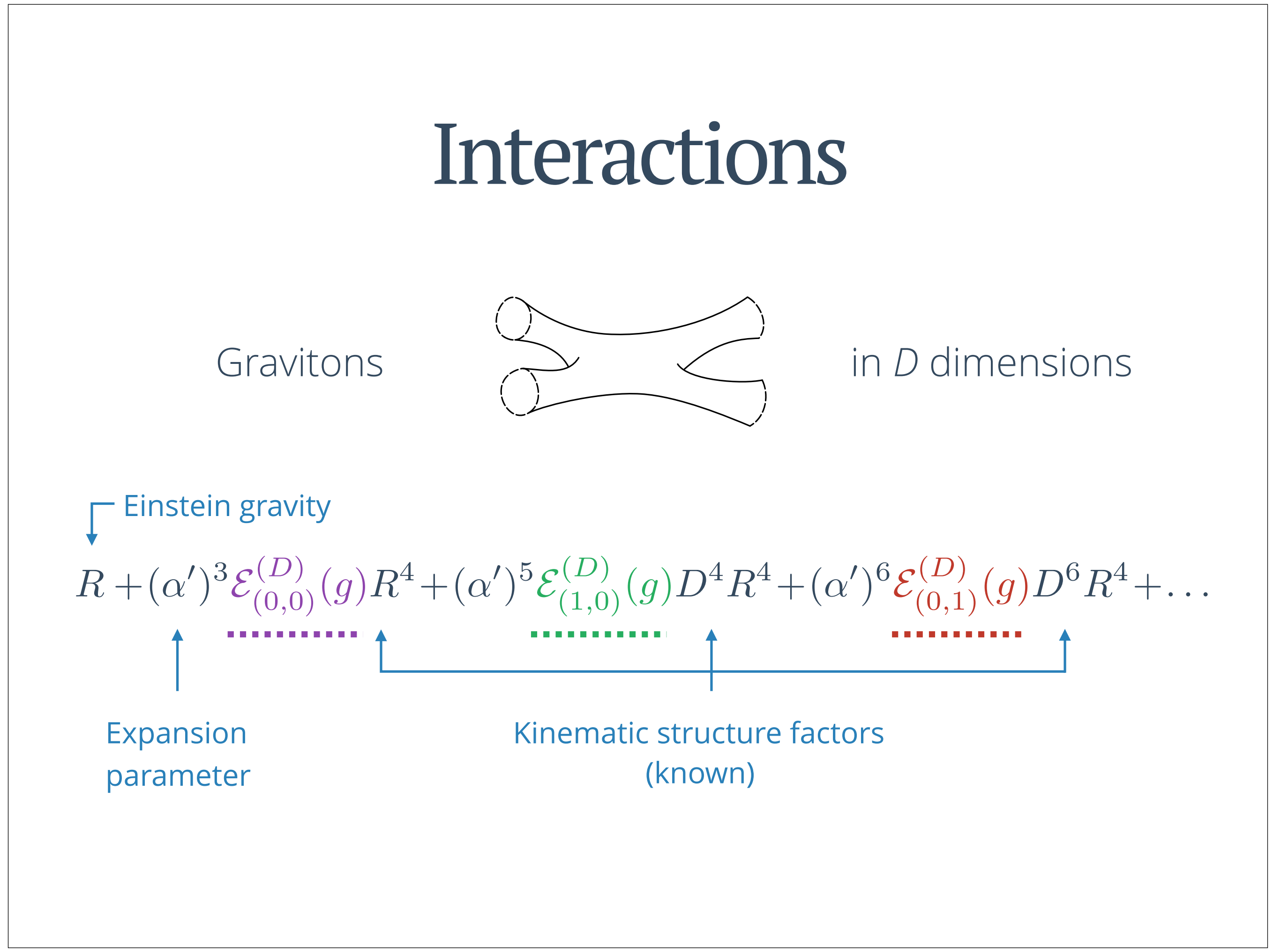

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$$\mathcal{M}_{\text{classical}} = G(\mathbb{R})/K$$

[Cremmer-Julia]

The coefficients are functions on a coset space G/maximal compact subgroup K called the moduli space.

The groups for different dimensions are shown in this table here and that can be visualized in this Dynkin diagram by adding simple roots in this order. Bourbaki labelling.

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8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$
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6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$
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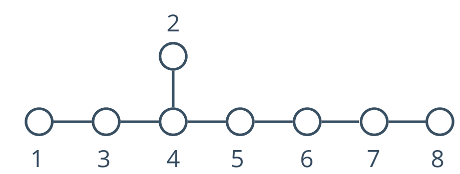

[Cremmer-Julia]

Note especially 10 dim and 5, 4, 3 dim

Moduli space

10 dimensions:

In ten dimensions the coset space is isomorphic to the upper half plane parametrized by the string coupling constant and a parameter called the axion.

For brevity we will denote the coefficient functions in ten dimensions as follows.

Moduli space

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$$\tau = \chi + i g_{\mathrm{s}}^{-1} \in \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\} \cong SL(2,\mathbb{R})/SO(2,\mathbb{R})$$

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U-duality

$G(\mathbb{R}) \circlearrowleft \mathcal{M}_{\text{classical}}$ classical symmetry

[Hull-Townsend]

The group G(R) also acts on the moduli space giving a symmetry of the classical theory.

However, string theory is a quantum theory so we have quantization of charges which breaks the classical symmetry to a discrete symmetry called U-duality and these are shown in this third column here.

These are the symmetry transformation that conserve the discrete charge lattice.

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8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$	$Spin(5,5;\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
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All observables are invariant under $G(\mathbb{Z})$

[Hull-Townsend]

The group G(R) also acts on the moduli space giving a symmetry of the classical theory.

However, string theory is a quantum theory so we have quantization of charges which breaks the classical symmetry to a discrete symmetry called U-duality and these are shown in this third column here.

These are the symmetry transformation that conserve the discrete charge lattice.

U-duality

D	$G(\mathbb{R})$	K	$G(\mathbb{Z})$
10	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
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$$\mathcal{E}^{(D)}_{(0,0)}(g),\ \mathcal{E}^{(D)}_{(1,0)}(g),\ \mathcal{E}^{(D)}_{(0,1)}(g)\ : G(\mathbb{Z})\backslash G(\mathbb{R})/K \to \mathbb{C}$$

Meaning, our coefficients are functions on this space

This looks a lot like automorphic forms…

Automorphic forms

An *automorphic form* is a smooth function $\varphi : G(\mathbb{R}) \to \mathbb{C}$ satisfying the following conditions

which are function on G that satisfy the following conditions:
A: they are U-duality invariant
B: K-finite (we will only consider spherical automorphic forms where this is trivially satisfied)
C: they are eigenfunctions to G-invariant differential operators (such as the laplacian)
D: they are of moderate growth

To be more precise the condition C can be specified as follows where Z is the center of the universal enveloping algebra, and then X here acts as a differential operator.

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And the growth condition means that they should grow as most as a polynomial.

Our coefficient functions are U-duality invariant and K-finite.
From computations in string theory using the diagrams with different genera i showed before, one can se that the coefficient functions also satisfy the growth condition.

But to answer C, we will have to study another symmetry of the theory: supersymmetry

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Supersymmetry constraints

bosons SUSY fermions

which is a symmetry relating bosons with fermions.

In ten dimension one obtains the following differential equations, where we see that the first two corrections satisfy the eigenfunction eq, meaning that they are automorphic forms.

However the third correction, gets an inhomogeneous RHS, and is thus not an eigenfunction and not an automorphic form in a strict sense.
We will come back to this in the outlook. The same pattern follows for lower dimensions.

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Eisenstein series

Since we have shown that E00 and E10 are automorphic forms, let us study a typical example of an automorphic form - a non-holomorphic Eisenstein series in ten dimensions i.e. on SL2.
They are constructed from a character \chi on the Borel subgroup which can be seen as the imaginary part of tau to some power of a complex number s.
The Eisenstein series is then defined as a sum over images for these characters - automatically giving an SL(2,Z) invariant function.

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$$\sigma_s(m) = \sum_{d\,|\,m} d^s$$

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Eisenstein series

$$E(s;\tau) = \sum_{\gamma\in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \chi(\gamma(\tau)) = \sum_{\gamma\in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \frac{\tau_2^s}{|c\tau+d|^{2s}} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$$\big(\Delta - s(s-1)\big)E(s;\tau) = 0$$

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$$E(s;\tau) = \tau_2^s + \frac{\xi(2s-1)}{\xi(2s)}\tau_2^{1-s} + \frac{2\tau_2^{1/2}}{\xi(2s)} \sum_{m\neq 0} |m|^{s-1/2}\,\sigma_{1-2s}(m) K_{s-1/2}(2\pi\,|m|\,\tau_2) e^{2\pi i m \tau_1}$$

Completed Riemann zeta function

$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$

Divisor sum

$\sigma_s(m) = \sum_{d\,|\,m} d^s$

Bessel function of the second kind

The Eisenstein series are eigenfunctions to the Laplacian with eigenvalues s(s-1), and since they are, by construction, invariant under SL(2,Z), they are periodic in the variable tau_1.
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[Green-Gutperle, Pioline, Green-Russo-Vanhove]

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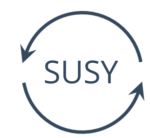

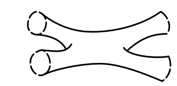

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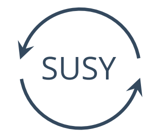

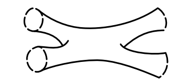

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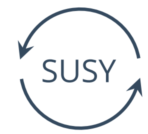

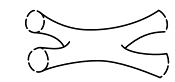

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$\mathcal{E}_{(0,1)}(\tau)$ as a sum over images $\sum_{B(\mathbb{Q})\backslash G(\mathbb{Z})}$ but not of a character χ

[Green-Miller-Vanhove, Kleinschmidt]

Extracting physical information

Expand Bessel function in g_s $\tau = \chi + i g_\mathrm{s}^{-1}$

[Green-Gutperle]

To connect back with physics, we can extract physical information from these functions be expanding in the string coupling constant.The first two terms, zero-mode, are perturbative in gs are exactly those that one obtains from string computations with the genus diagrams.
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In the exponential we see, what is called the instanton action, which comes from certain solutions to the Einstein equation called instantons. These are the objects that give rise to non-perturbative effects. They are labeled by the mode m which we call instanton charges.

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In front of the exponential we have an instanton measure counting the number of states for a given instanton charge m, which we find is the number of ways m can be factorised into two integers. These integers have the physical interpretation of being the wrapping number and charge of a T-dual D-particle to our D-instanton.

Indeed a wealth of information and powerful predictions - for example, we see that there are only two genus diagrams contributing to this interaction - the higher genus diagrams have to cancel! And this has later been checked in a lot of cases.

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Perturbative
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arithmetic information
p-adic part

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wrapping number of a T-dual D-particle and charge

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Lower dimensions

We would now like to do the same analysis for lower dimensions where we recall that we had the following table of groups and similar coefficients functions on G

And one can show that the coefficient functions are also Eisenstein series.

Lower dimensions

D	$G(\mathbb{R})$	K	$G(\mathbb{Z})$
10	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$SL(2,\mathbb{Z}) \times \mathbb{Z}_2$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$	$Spin(5,5;\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

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Parabolic subgroups

Σ choice of simple roots $\quad\quad \langle\Sigma\rangle$ generated root system

Before that, let me quickly go through some definitions to get us all on the same page.

We need to define parabolic subgroups which are specified by a choice of simple roots - a set Sigma. Let <Sigma> be the generated subroot system of these simple roots.

And g_alpha the usual definition.

Then the lie algebra p is constructed from the Cartan subalgebra + all the positive roots together with the generated root system.

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Corresponding group P

Let us visualize this for SL(4) with the choice of Sigma being only the first simple root.

Then the subgroup L looks like this, with the generated root system labelled in red. And U with the remaining positive roots. P is then the product of the two.

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$$L = \left\{ \begin{pmatrix} \boxed{\begin{matrix} * & * \\ * & * \end{matrix}} & & \\ & \boxed{*} & \\ & & \boxed{*} \end{pmatrix} \right\} \qquad U = \left\{ \begin{pmatrix} \boxed{\begin{matrix} 1 & \\ & 1 \end{matrix}} & \begin{matrix} * & * \\ * & * \end{matrix} \\ & \begin{matrix} \boxed{1} & * \\ & \boxed{1} \end{matrix} \end{pmatrix} \right\}$$

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Then the subgroup L looks like this, with the generated root system labelled in red. And U with the remaining positive roots. P is then the product of the two.

Parabolic subgroups

$$\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u} \qquad \mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \langle \Sigma \rangle} \mathfrak{g}_\alpha \qquad \mathfrak{u} = \bigoplus_{\alpha \in \Delta(\mathfrak{u})} \mathfrak{g}_\alpha$$

Corresponding group P

$G = SL(4)$  $\Sigma = \{\alpha_1\}$

$$L = \left\{ \begin{pmatrix} * & * & & \\ * & * & & \\ & & * & \\ & & & * \end{pmatrix} \right\} \qquad U = \left\{ \begin{pmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right\}$$

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$$P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \right\} = LU$$

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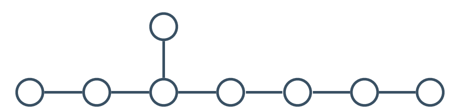

If we choose, here in another example, to include all simple roots but one, the parabolic subgroup is called maximal.

On the other hand, if we don't include any at all, it becomes the Borel subgroup, also called the minimal parabolic. Each with their respective decompositions denoted like this.

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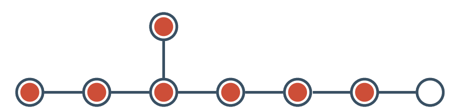

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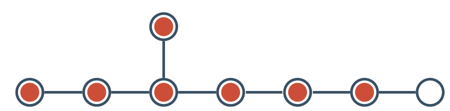

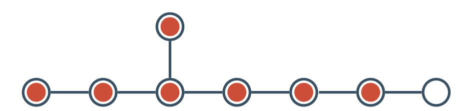

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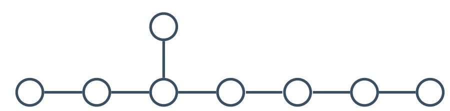

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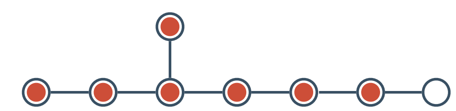

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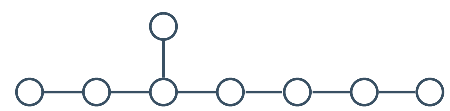

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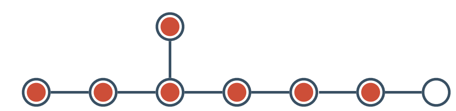

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$$B = NA \qquad P = LU$$

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Eisenstein series

Let $\chi_P : P(\mathbb{Z})\backslash P(\mathbb{R}) \to \mathbb{C}^\times$ be a multiplicative character determined by its restriction on L and trivially extended to all of G.

Eisenstein series for higher rank groups are then constructed from a parabolic subgroup P and a multiplicative character chi on this, which is determined by it restriction on L and trivially extended to all of G.

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Fourier expansion

Expand in different directions $\longleftrightarrow$ Unipotent subgroup U

$$u = \prod_{\alpha\in\Delta^{(1)}(\mathfrak{u})} \exp(u_\alpha E_\alpha) \mapsto \exp(2\pi i \sum_{\alpha\in\Delta^{(1)}(\mathfrak{u})} m_\alpha u_\alpha)$$

Since we have a larger group we may also Fourier expand in several different directions, which amounts to a choice of unipotent subgroup U and such a group can be obtained from a choice of another parabolic subgroup P.

Let psi be a multiplicative character on U where U(1) is the circle parametrized by integer charges m. Where Ealpha are the positive Chevalley generators.

Fourier expansion

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Choice of another parabolic P

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$$F_U(\chi, \psi; g) = \int\limits_{U(\mathbb{Z})\backslash U(\mathbb{R})} E(\chi, ug)\overline{\psi(u)}\, du$$

The F coeff is then defined as this integral over U of the Eisenstein series and the complex conjugate of the character.

The original function is obtained by summing over Fourier modes, which we usually split into a constant term with trivial character and the remaining modes.

However, one can show that FU depends on u in a trivial way by multiplication of the character \psi(u) and since \psi is multiplicative on U this means that the above sum over Fourier modes can only capture the abelian part of U. If U is non-abelian one has to include Fourier coefficients on the commutator subgroups of U as well.

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Separated because the methods for computing them are very different

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Some $m_\alpha = 0$ degenerate

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Choice of unipotent subgroup U $\longleftrightarrow$ Study different perturbative and non-perturbative effects

[Green-Miller-Vanhove]

To connect back to physics the different unipotent subgroups we can Fourier expand in allows us to study different perturbative and non-perturbative effects string theory. Here are three important examples.

First, the string perturbation limit which we have studied before when the string coupling is small. This amounts to an expansions wrt this maximal parabolic subgroup.
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Such Fourier coefficients are, however, difficult to compute which is why we turn to the adelic framework.

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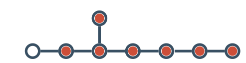
- String perturbation limit
  D-instantons | NS5-instantons



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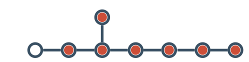

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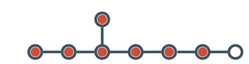

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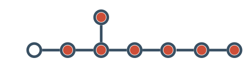

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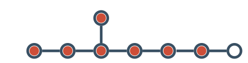

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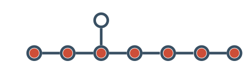

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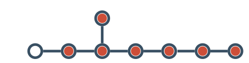

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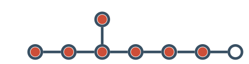

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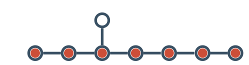

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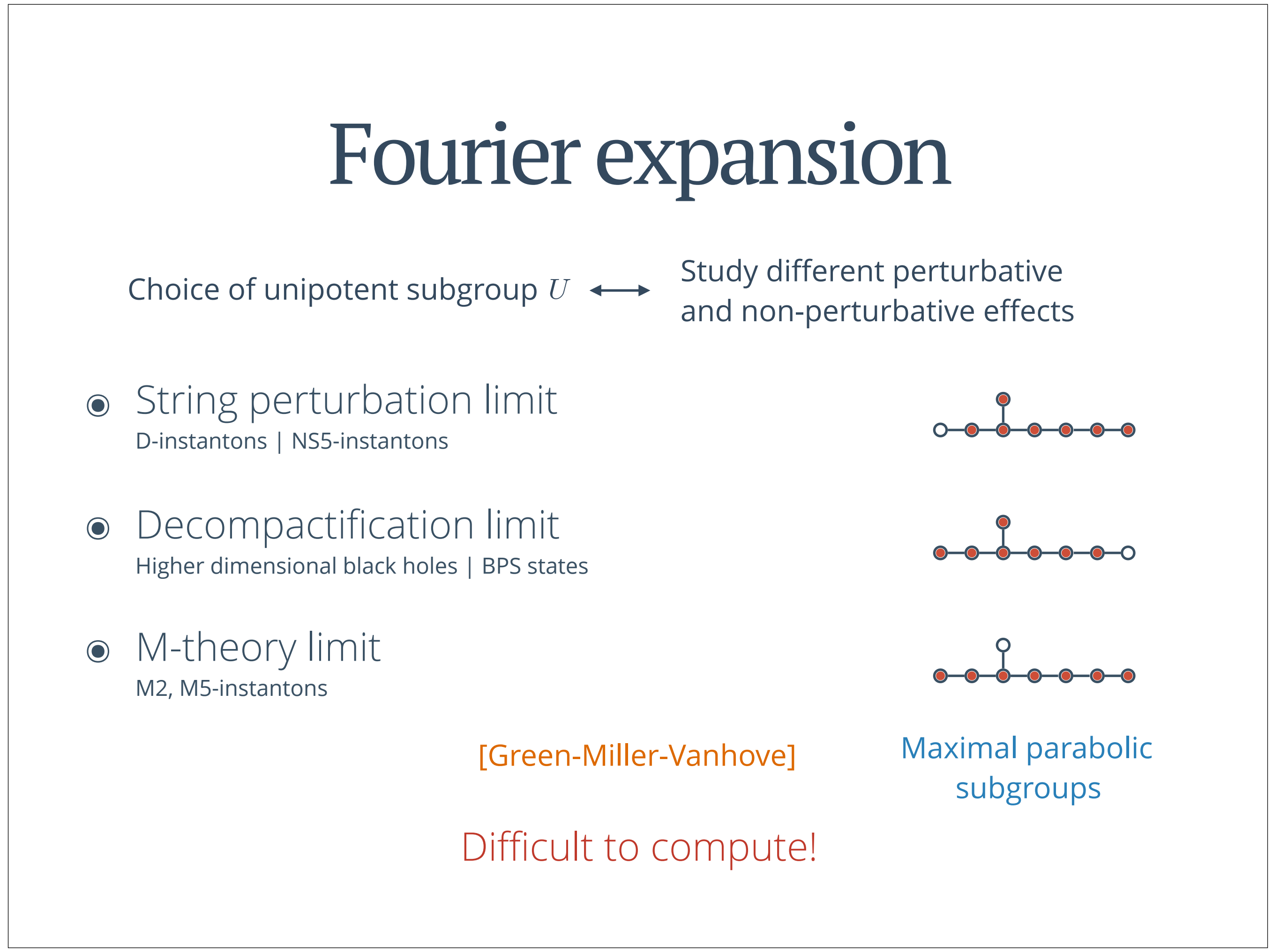

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Adelic framework

> *An efficient, but abstract, way to approach the subject of automorphic forms is by the introduction of adeles, rather ungainly objects that nevertheless, once familiar, spare much unnecessary thought and many useless calculations.*

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$$\mathcal{E}^{(D)}_{(0,0)}(g),\ \mathcal{E}^{(D)}_{(1,0)}(g),\ \mathcal{E}^{(D)}_{(0,1)}(g)\ : G(\mathbb{Z})\backslash G(\mathbb{R})/K \to \mathbb{C}$$

So we lift our coefficient function to the adeles of the rationals
With G(A) looking like this and the maximal compact subgroup KA like this.

Using strong approximation we can then study the coefficient functions on this space instead.

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Lift to the adeles [arXiv:1511.0465 §4.2.2]

$$\mathbb{A} = \mathbb{A}_{\mathbb{Q}} \qquad G(\mathbb{A}) = G(\mathbb{R}) \times \prod_{p \text{ prime}}{}' G(\mathbb{Q}_p) \qquad K_{\mathbb{A}} = K \times \prod_{p \text{ prime}} G(\mathbb{Z}_p)$$

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$$\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E(\chi; ug)\overline{\psi_{\mathbb{R}}(u)}\, du \qquad \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(\chi; ug)\overline{\psi_{\mathbb{A}}(u)}\, du$$

$$m_\alpha \in \mathbb{Z} \qquad m_\alpha \in \mathbb{Q}$$

Automorphic representations

$G(\mathbb{A}) \circlearrowright$ Space of automorphic forms*

* With some subtleties described in [arXiv:1511.0465 §5.4]

The group G(A) acts on the space of automorphic forms with some subtleties described in the review.

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Computing adelic Fourier coefficients

[arXiv:1511.0465 §8-9]

Whittaker vectors

Now turning to the computation of adelic Fourier coefficients.
In our review we have gathered and extended methods for computing Whittaker vectors. First the constant term using Langland's constant term formula. Then unramified Whit vec using the Casselman-Shalika formula. And this allows us to then compute generic and lastly, degenerate Whit vec.
Important to note here is that, the more degenerate a Whit vec is - the easier it actually becomes to compute. A maximally degenerate Whit vec looks like and SL(2) Whit vec.

In the paper sharing the title of this talk, we compute Fourier coefficients in terms of these (known) Whittaker vectors of automorphic forms in small representations.

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Fourier coefficients

In terms of Whittaker vectors
Simplify drastically for small representations

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Previous results

[Miller-Sahi]

It all started with a paper from Miller-Sahi that got us really excited. They showed that for …

This seemed very promising for our goal. The only thing we needed now was an explicit formula for computing our Fourier coefficients, so we started working on this using the same tools that Miller and Sahi used.

Previous results

For $G = E_6, E_7$, an automorphic form $\varphi \in \pi_{\min}$ is completely determined by maximally degenerate Whittaker vectors

$$W_N \quad \text{with only one} \quad m_\alpha \neq 0$$

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Character variety orbits

$$F_U(\chi,\psi;g) = \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(\chi;ug)\overline{\psi(u)}\,du \qquad P = LU$$

Namely character variety orbits

If we take an element gamma in L(Q) one can show that a Fourier coeff with translated argument gamma g equals the Fourier coeff with a conjugate character \psi gamma.

This gives us what is called character variety orbits, which are more conveniently described by identifying psi with omega in the dual of u. This means that we only have to compute one Fourier coeff per orbit.

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$\psi \longleftrightarrow \omega \in \mathfrak{u}^*$ $\quad$ Adjoint action under $L(\mathbb{Q})$

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$$L(\mathbb{Q}) \cdot \omega$$

We will use a coarser description for the character variety orbits using complex orbits of the whole of G, which we will simply call nilpotent orbits.

Additionally, to each automorphic representation, one can associate a so called special nilpotent orbit. Which will give us a connection between Fourier coefficients and representations.

Let us first study some properties of nilpotent orbits.

Character variety orbits

Coarser description

Character variety orbits $\longrightarrow$ Nilpotent orbits

$L(\mathbb{Q}) \cdot \omega$ $\qquad$ $\mathcal{O} = \{gXg^{-1} \mid g \in G(\mathbb{C})\}$

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Organizing orbits

[Collingwood-McGovern]

For $SL(n)$, orbits can be identified with partitions of n

where we see here the trivial, minimal and ntm orbits corresponding to the trivial, minimal and ntm representations.

More generally, a partial ordering can be obtained from inclusion wrt Zarisky closure.

Another way of labeling is Bala-Carter label based on distinguished parabolic subalgebras.

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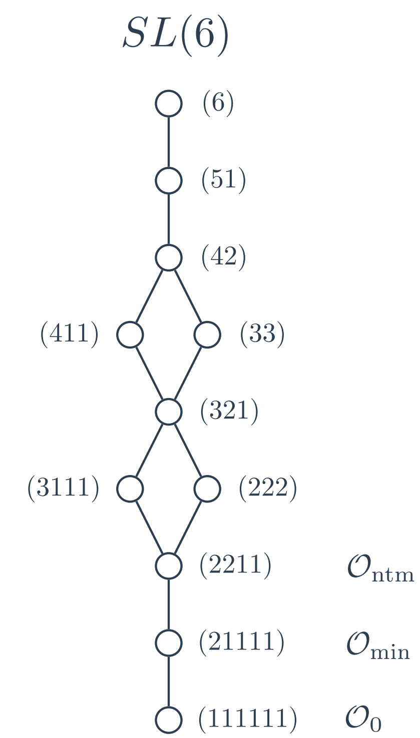

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Coming back to Fourier coeff and automorphic representations, if we let the Eisenstein series E(chi) be in the representation pi.

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$F_U(\chi, \psi; g)$ vanishes unless $\psi \in \mathcal{O} \leq \mathcal{O}_\pi$

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

Then, there's a theorem from Matumoto, Maeglin-Waldspurger, extended by all these people which tells us that a Fourier coeff of this Eisenstein series vanishes unless psi is in an orbit less or equal than O_pi

Meaning that automorphic forms in small automorphic representations have few non-vanishing Fourier coefficients.

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Small automorphic representations have few non-vanishing Fourier coefficients

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Main results

Goal: find expressions for Fourier coefficients
in terms of Whittaker vectors (known)
using vanishing properties given π

[arXiv:1412.5625]

We started with studying G = SL(3) and SL(4).

Although we expect our results to hold for arbitrary simply-laced groups

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after discussing with Gourevitch and Sahi, the proper generalisation seems to be strongly orthogonal roots instead of commuting roots.

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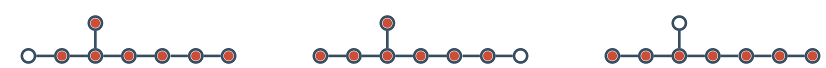
Fourier coefficients on maximal parabolic subgroups



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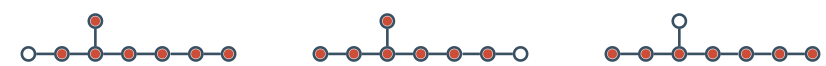
Fourier coefficients on maximal parabolic subgroups



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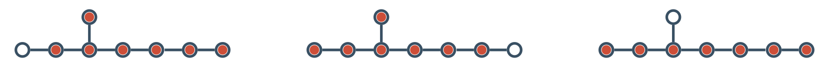

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$$\mathcal{E}^{(D)}_{(0,0)} \in \pi_{\min}$$

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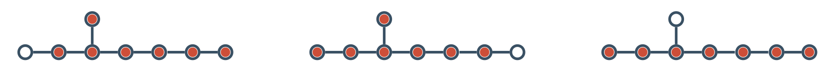

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[arXiv:1412.5625]

We showed that a maximal parabolic F coeff in min rep = a single maximally degenerate Whittaker vector with translated argument depending on \psi

We proved this only for SL(3), and SL(4), but suspected that it holds for arbitrary simple simply laced lie groups as well.

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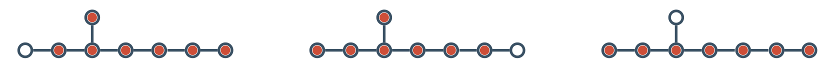

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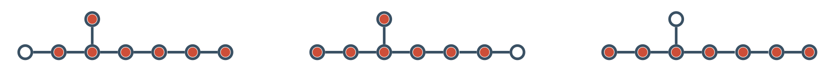

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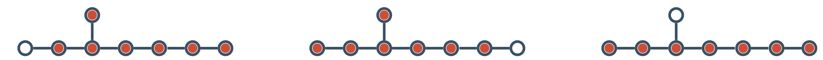

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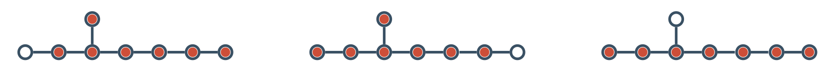

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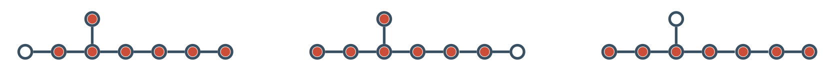

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Local spherical vectors for E_6, E_7, E_8

So we wanted to test the corresponding statement for E6, E7 & E8 by studying so called local spherical vectors.

The embedding of the LOCAL minimal representation in the induced representation of \psi is of multiplicity one and the unique local spherical vectors f have been computed for several groups and subgroups U at both the archimedean and non-archimedean places using techniques from representation theory.

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$$\pi_{\min,p} \subset \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_{\min,p} \hookrightarrow \mathrm{Ind}_{U(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \psi_{U,p}$$

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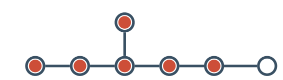
For example: E_7 with U from 

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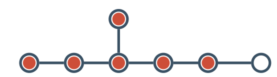
For example: E_7 with U from 

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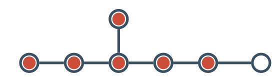
For example: E_7 with U from 

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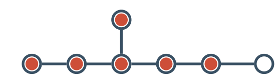
For example: E_7 with U from 

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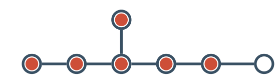
For example: E_7 with U from 

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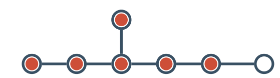
For example: E_7 with U from 

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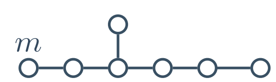
ψ_N : 

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Complete agreement for E_6, E_7, E_8 in both abelian and Heisenberg realisations

We find complete agreement for E6, E7 and E8 for both the abelian and Heisenberg realisations corresponding to different unipotent subgroups U.

This is strong evidence for that the above relation can be generalized to higher rank groups.

Outlook

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HG, Axel Kleinschmidt, Dmitry Gourevitch, Siddhartha Sahi, Daniel Persson

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Work in progress

CHALMERS
UNIVERSITY OF TECHNOLOGY

Thank you!

Henrik Gustafsson

Number Theory Seminar

Rutgers 2016

hgustafsson.se

Backup slides

Automorphic representation

$$[\pi_f(h_f)\varphi](g) = \varphi(g(\mathbb{1}; h_f)) \qquad h_f \in G_f$$

$$[\pi_{K(\mathbb{R})}(k_\infty)\varphi](g) = \varphi(g(k_\infty; \mathbb{1})) \qquad k_\infty \in K(\mathbb{R})$$

$$[\pi_{\mathfrak{g}}(X)\varphi](g) = \tfrac{d}{dt}\varphi(ge^{tX})|_{t=0} \qquad X \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$$

K-finiteness

$$\dim_{\mathbb{C}}\Big(\operatorname{span}\{\varphi(gk) \mid k \in K_{\mathbb{A}}\}\Big) \leq \infty\,.$$

Whittaker vectors

$$G(\mathbb{Q}) = \bigcup_{w \in \mathcal{W}} B(\mathbb{Q}) w B(\mathbb{Q}) \qquad\qquad N^{(w)}(\mathbb{A}) = \prod_{\substack{\alpha > 0 \\ w\alpha < 0}} N_\alpha(\mathbb{A})$$

$$W_N(\chi; a) = \sum_{\gamma \in B(\mathbb{Q}) \backslash G(\mathbb{Q})} \int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} \chi(\gamma n a) \overline{\psi(n)} \, dn = \sum_{w \in \mathcal{C}_\psi} F_w(\chi; a)$$

$$F_w(\chi; a) = \int_{N^{(w)}(\mathbb{A})} \chi(w n a) \overline{\psi(n)} \, dn$$

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Whittaker models

$$\mathrm{Ind}_{N(\mathbb{A})}^{G(\mathbb{A})}\psi = \left\{ W_\psi : G(\mathbb{A}) \to \mathbb{C} \,\middle|\, W_\psi(ng) = \psi(n) W_\psi(g),\ n \in N(\mathbb{A}) \right\}.$$